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Graph-like combinatorial structures in $(r, 1)$-designs

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Abstract

This paper is a short report on those $(r, 1)$-designs which are not discussed explicitly in Gropp (1994). There are altogether 974 $(r, 1)$-designs with at most 12 points. Most of them are configurations (v_r, b_k). Moreover, as further examples occur regular graphs and also some previously unknown and more irregular structures which are discussed in this paper.

1. Introduction

In this paper the same notation is used as in [1]. Especially the definitions of linear space, $(r, 1)$-design, configuration (v_r, b_k), and the configuration graph are given there explicitly.

Section 2 discusses the $(r, 1)$-designs of type G+, i.e. those which can be interpreted as regular graphs with additional properties. The main section is Section 3 which discusses the most irregular $(r, 1)$-designs, i.e. those of type F. These designs are probably the smallest examples of a large set of combinatorial structures which cannot be described by means of previously known structures as configurations, graphs, etc. That is why I decided to describe them in a special paper.

Table 2 of [1] shows that at least for small values of v the $(r, 1)$-designs are very rare among all linear spaces and, on the other hand, 'nearly all' $(r, 1)$-designs are configurations. However, it is also worth to study those 'few' $(r, 1)$-designs which are not configurations. In [1, Section 5] the 974 $(r, 1)$-designs with at most 12 points are classified into certain types. In this paper the types F and G+ will be discussed explicitly. The other types are discussed in [1].

Those point types which are used in the following are listed in Table 1. The point type describes on how many lines of a certain size the point lies.

For example, point type I_7 means that such a point lies on 2 4-lines, on 2 3-lines and on $r-4$ 2-lines.

Now for a certain value of v and a certain value of r all point type distributions can be computed as shown in [1].

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Table 1

Point types	r_8	r_7	r_6	r_5	r_4	r_3	r_2
C_5			0	0	2	0	$r-2$
D_5			0	0	1	2	$r-3$
E_5			0	0	0	4	$r-4$
D_6		0	0	1	0	2	$r-3$
E_6		0	0	0	2	1	$r-3$
F_6		0	0	0	1	3	$r-4$
G_6		0	0	0	0	5	$r-5$
H_7	0	0	0	0	3	0	$r-3$
I_7	0	0	0	0	2	2	$r-4$
J_7	0	0	0	0	1	4	$r-5$

2. Regular graphs with additional properties (Type G+)

There are altogether 20 $(r, 1)$-designs classified as Type G+, (see Table 2).

These $(r, 1)$-designs can be interpreted either as regular graphs with one-factorization or as bipartite regular graphs. The explicit presentation of the designs can be found in [1].

2.1. Regular graphs with one-factorization

2.1.1. $v=12, r=7, d_5=12, n_4=3, n_3=8, n_2=24$

The eight 3-lines form a cubic graph with 8 vertices. Each 4-line is a set of 4 disjoint edges; hence the three 4-lines together are a one-factorization of the cubic graph. Such structures have been enumerated by Rosa and Stinson [2]. The eight nonisomorphic designs are exhibited in [1, Subsection 6.3.2].

2.1.2. $v=12, r=6, e_6=12, n_4=6, n_3=4, n_2=18$

Here the 4-lines form a 4-regular graph with 6 vertices. This is unique since it is the complement of a 1-regular graph with 6 vertices. The 3-lines are a one-factorization. Such a structure is unique (see [2]).

Table 2

Number	v	r	Point type distribution	Section
8	12	7	$d_5=12$	2.1.1
1	12	6	$e_6=12$	2.1.2
1	11	5	$e_6=10, g_6=1$	2.1.3
3	12	6	$e_6=3, f_6=6, g_6=3$	2.1.4
1	12	5	$i_7=12$	2.1.5
2	12	6	$d_6=10, g_6=2$	2.2.1
2	10	5	$d_5=8, e_5=2$	2.2.2
2	11	5	$f_6=8, g_6=3$	2.2.3

2.1.3. $v=11$, $r=5$, $e_6=10$, $g_6=1$, $n_4=5$, $n_3=5$, $n_2=10$

The 4-lines form a 4-regular graph with 5 vertices, i.e. the complete graph K_5. All the 3-lines contain point 11 and 2 other points (i.e. 2 edges which do not meet in K_5). Such a structure is called a near-factorization of K_5 and is unique.

2.1.4. $v=12$, $r=6$, $e_6=3$, $f_6=6$, $g_6=3$, $n_4=3$, $n_3=12$, $n_2=12$

The points of $\{10, 11, 12\}$ occur 5 times, those of $\{4, 5, 6, 7, 8, 9\}$ occurs 3 times on a 3-line. Hence w.l.o.g. there are three 3-lines $\{1, 10, 11\}$, $\{2, 10, 12\}$, $\{3, 11, 12\}$. The three 4-lines must be $\{1, 2, 4, 5\}$, $\{1, 3, 6, 7\}$, $\{2, 3, 8, 9\}$. Let $\{4, 5, 6, 7, 8, 9\}$ be the set of vertices of a graph. The three further 3-lines through point 10 form a 1-factor of this graph. The same holds for the lines through 11 and through 12. So these 9 remaining 3-lines can be described by the 3-colouring of the complement of a 6-cycle which is unique. The 3 colours also occur as 3 edges of the 6-cycle which yields 3 nonisomorphic designs which are exhibited in [1, Subsection 6.3.6].

2.1.5. $v=12$, $r=5$, $i_7=12$, $n_4=6$, $n_3=8$, $n_2=6$

The 4-lines form a 4-regular graph with 6 vertices which is unique. The eight 3-lines are then uniquely determined. They can be regarded as 2 orthogonal one-factorizations.

2.2. Bipartite regular graphs

2.2.1. $v=12$, $r=6$, $d_6=10$, $g_6=2$, $n_5=2$, $n_3=10$, $n_2=16$

The two 5-lines are $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$. Hence each 3-line contains only 2 points of $\{1, \ldots, 10\}$. The 5-lines through point 11 and those through point 12 can be regarded as two 1-factors. Their union is a 2-regular bipartite graph with partition sets $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$, i.e. all occurring cycles are even. This is either a (Hamiltonian) 10-cycle or the union of a 6-cycle and a 4-cycle.

2.2.2. $v=10$, $r=5$, $d_5=8$, $e_5=2$, $n_4=2$, $n_3=8$, $n_2=9$

This is the same problem as above with 8 instead of 10 vertices. The solution is either a (Hamiltonian) 8-cycle or the union of two 4-cycles.

2.2.3. $v=11$, $r=5$, $f_6=8$, $g_6=3$

This is a 3-regular bipartite graph with partition sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. There are exactly 2 such graphs.

3. Nonregular graphs (Type F)

The following 43 $(r, 1)$-designs are classified as Type F, (see Table 3).

In the following the construction of these 43 designs is described shortly. In some cases some detailed considerations are omitted. My aim is to describe some typical

Table 3

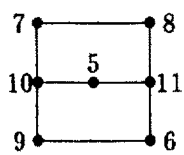
Number	v	r	Point type distribution	Section
7	11	6	$d_5=4$, $e_5=7$	3.1
8	12	7	$c_5=1$, $d_5=10$, $e_5=1$	3.5
17	12	7	$c_5=2$, $d_5=8$, $e_5=2$	3.2
10	12	6	$e_6=2, f_6=8$, $g_6=2$	3.3
1	12	5	$h_7=2$, $i_7=8$, $j_7=2$	3.4

examples, their special attitudes and some methods to construct these designs. The investigation of larger designs will decide which of these methods can be generalized and whether the designs described here are the typical representatives of this new class of combinatorial structures. The designs are exhibited explicitly in [1].

3.1. $v=11$, $r=6$, $d_5=4$, $e_5=7$, $n_4=1$, $n_3=12$, $n_2=13$

The unique 4-line is $\{1,2,3,4\}$. There are four 3-lines which contain 3 points of type E_5 and one 2-line which contains 2 points of type E_5, say $\{10,11\}$. Both these points occur a second time on a 2-line; hence they are connected with the 3 other points in $\{1,2,3,4\}$ on three 3-lines and occur exactly once on the four 3-lines with only points of type E_5. Similar arguments imply that the points of $\{5,6,7,8,9\}$ occur twice in these four 3-lines. Up to isomorphism these four lines are $\{5,6,7\}$, $\{5,8,9\}$, $\{6,8,10\}$, $\{7,9,11\}$.

The task is now to determine the remaining eight 3-lines. This can be interpreted in the following way. Define a graph on the 7 vertices 5, 6, 7, 8, 9, 10, 11 in the following way (similar to the configuration graph of a configuration): Two vertices are connected by an edge iff they are not collinear in one of the four lines above. The resulting graph is the following $\{5,10\}$, $\{5,11\}$, $\{6,9\}$, $\{6,11\}$, $\{7,8\}$, $\{7,10\}$, $\{8,11\}$, $\{9,10\}$.



Equivalent to the problem of finding the remaining 8 lines of the $(r,1)$-design is the problem of colouring the edges of this graph with 4 colours such that each colour occurs twice and only edges with different colours meet each other. The colours 1, 2, 3, 4 correspond to the points of the $(r,1)$-design in such a way that the design contains the line $\{c,a,b\}$ if the edge $\{a,b\}$ is coloured with colour c.

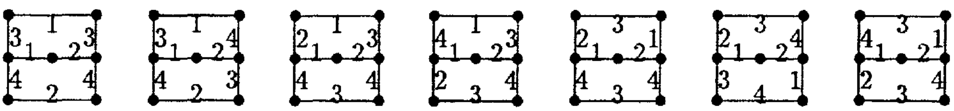
The above graph has as distinguished edges the 2 'short edges' 5, 10 and 5, 11 as well as the 2 'long edges' 6, 9, and 7, 8. w.l.o.g. colour the edge 5, 10 with colour 1 as well as the edge 5, 11 with colour 2.

The first case then is that the 2 long edges are also coloured with colours 1 and 2. This yields the solutions no. **1** and no. **2**. Either the two neighbouring edges of the long edges are coloured equally or they are coloured with different colours.

The second case is that one of the long edges is coloured with colour 1 and the other one with a new colour, say 3. This yields solutions no. **3** and no. **4**. Either the two edges coloured with 4 are neighbours of the long edge with colour 3 or they are not.

The third case is that the two long edges are coloured equally but not with colour 1 or 2. This yields solutions no. **5** and no. **7**. Again the two edges coloured with 4 are neighbours of the same long edge or they are not.

The last case where the two long edges are coloured with colours 3 and 4 leads to the unique solution no. **7**.



no. **1** no. **2** no. **3** no. **4** no. **5** no. **6** no. **7**

3.2. $v=12$, $r=7$, $c_5=2$, $d_5=8$, $e_5=2$, $n_4=3$, $n_3=8$, $n_2=24$

In the following four subcases will occur. They will be called Ia, Ib, IIa and IIb and will contain 4, 2, 2, and 9 solutions respectively. The subcases are characterized by the eight 3-lines. Afterwards it is necessary to determine the three 4-lines. One of them contains both points 1 and 2. The other two contain one of them each and two further points in $\{3,4,5,6,7,8,9,10\}$.

Case I: If the points 11 and 12 are on different 3-lines then they both determine a 1-factor in the graph with the vertices 3, 4, 5, 6, 7, 8, 9, 10 by drawing an edge between two vertices if they are on a line together with 11 or 12 respectively.

Subcase Ia: If the union of these two 1-factors is a cycle of length 8 (a Hamiltonian cycle), the eight 3-lines are $\{3,4,11\}$, $\{5,6,11\}$, $\{7,8,11\}$, $\{9,10,11\}$, $\{4,5,11\}$, $\{6,7,11\}$, $\{8,9,11\}$, $\{3,10,11\}$. In order to determine the 4-lines consider the 8-cycle mentioned above and classify the possible 4-lines according to the distances of the 3 points on the lines through 1 and 2 in this 8-cycle. These can be twice 4-2-2, once 4-2-2 and once 3-3-2, and twice 3-3-2 in two different ways.

Subcase Ib: If the union of these two 1-factors are 2 cycles of length 4, the eight 3-lines are $\{3,4,11\}$, $\{5,6,11\}$, $\{7,8,11\}$, $\{9,10,11\}$, $\{4,5,11\}$, $\{3,6,11\}$, $\{8,9,11\}$, $\{7,10,11\}$. The 4-line through the points 1 and 2 either contains 2 points of the same 4-cycle, say 3 and 5 (this leads to solution no. **5**) or 1 point of each 4-cycle, say 6 and 10 (this leads to solution no. **6**).

Case II: If the points 11 and 12 are connected on a 3-line w.l.o.g. five of the eight 3-lines are $\{3,11,12\}$, $\{4,5,11\}$, $\{6,7,11\}$, $\{8,9,11\}$, $\{3,9,10\}$. By only considering the vertices 4, 5, 6, 7, 8, 9, 10 the 3-lines define 7 edges in this set.

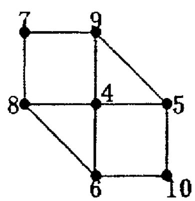
Subcase IIa: If these 7 edges form the union of a 3-cycle and a 4-cycle the remaining 3-lines are $\{4,7,12\}$, $\{5,6,12\}$, $\{8,10,12\}$. The points 8, 9, and 10 must be on 3 different 4-lines. Either point 8 is on the line through 1 and 2 (this leads to solution no. **7**) or one of the points 9 or 10, say 10, is on this line (this leads to solution no. **8**).

Subcase IIb: If the 7 edges mentioned above form a 7-cycle the remaining 3-lines are $\{4,6,12\}$, $\{5,8,12\}$, $\{7,10,12\}$. The 4-lines are determined similarly as in subcase Ia according to the distances of the points on this 7-cycle and yields the 9 solutions no. **9** to no. **17**.

3.3. $v=12$, $r=6$, $e_6=2$, $f_6=8$, $g_6=2$, $n_4=3$, $n_3=12$, $n_2=12$

The three 4-lines are chosen to be $\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{2,8,9,10\}$. The points 11 and 12 are connected on a 3-line. Otherwise e.g. point 1 would be twice on a 3-line. Hence there are four 3-lines containing point 11, four 3-lines containing point 12, and one 3-line containing 11 and 12. The 3 remaining 3-lines are $\{3,5,8\}$, $\{3,6,9\}$, $\{4,7,10\}$. The third point on the line through 11 and 12 can be 3 (Case II) or not 3 (Case I). Choose point 1 for Case I. In order to find the other eight 3-lines the 2 cases are discussed separately.

Case I: w.o.l.g. point 2 is on a line through 11 and point 3 is on a line through 12. Consider the graph on the set $\{4,5,6,7,8,9,10\}$ with those 10 edges which are not yet covered by the previously constructed lines. This graph can be realized as a 6-cycle $(5,9,7,8,6,10,5)$ with a 'central vertex' 4 which is connected to 5, 9, 8, and 6.



All possibilities for the lines through 11 are (since $(5,6)(8,9)$ is an automorphism of the already constructed lines)

M: $\{2,5\}$, $\{4,8\}$, $\{7,9\}$, $\{6,10\}$, N: $\{2,5\}$, $\{4,9\}$, $\{7,8\}$, $\{6,10\}$,

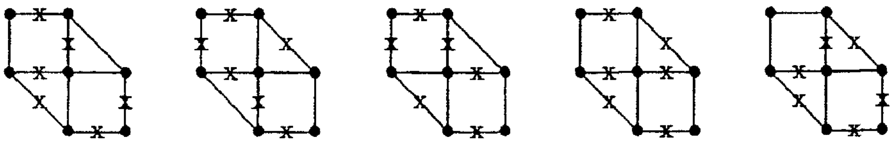
and Q: $\{2,7\}$, $\{4,8\}$, $\{5,9\}$, $\{6,10\}$,

and for those through point 12 are

I: $\{3,7\}$, $\{4,8\}$, $\{5,9\}$, $\{6,10\}$, J: $\{3,10\}$, $\{4,5\}$, $\{6,8\}$, $\{7,9\}$,

K: $\{3,7\}$, $\{4,9\}$, $\{5,10\}$, $\{6,8\}$, L: $\{3,10\}$, $\{4,6\}$, $\{5,9\}$, $\{7,8\}$.

There are exactly 5 possible combinations: MK(no. **1**), ML(no. **2**), NJ(no. **3**), QJ(no. **4**), and QK(no. **5**).



no. 1 no. 2 no. 3 no. 4 no. 5

In order to show that they are nonisomorphic they can be distinguished in the following way. Those 6 edges are labelled which describe lines through 11 or 12 apart from the lines through 2 and 3. The 5 subgraphs which are obtained are nonisomorphic.

Case II : W.o.l.g. point 1 is on a line through 11 and point 2 is on a line through 12. Again the same graph as above is considered. Now there are the same possibilities for the lines through 12 as above (M, N, and Q) and a similar list for the lines through 11. Again there are exactly 5 nonisomorphic solutions: no. **6** to no. **10**.

3.4. $v=12$, $r=5$, $h_7=2$, $i_7=8$, $j_7=2$, $n_4=6$, $n_3=8$, $n_2=6$

The points of type H_7 do not occur on a 3-line, those of type J_7 do not occur on a 2-line. Hence all connections of points of these types must occur on 4-lines. The points of type J_7 lie only on one 4-line and must be connected to both points of type H_7. Hence one 4-line is $\{1,2,11,12\}$ and there is one 4-line without points 1 and 2, say $\{7,8,9,10\}$. W.l.o.g. the other four 4-lines are $\{1,3,4,7\}$, $\{1,5,6,8\}$, $\{2,3,5,9\}$, $\{2,4,6,10\}$.

The eight 3-lines are constructed uniquely since 7, 8, 9, 10 as well as 11 and 12 must not occur on the same line. Hence their is a unique $(r,1)$-design.

3.5. $v=12$, $r=7$, $c_5=1$, $d_5=10$, $e_5=1$, $n_4=3$, $n_3=8$, $n_2=24$

The 4-lines are $\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{8,9,10,11\}$. Now regard the four 3-lines through point 12. Each of them contains one point of $\{8,9,10,11\}$ and one point of $\{2,\dots,7\}$. Important is the fact, however, that not all points of $\{2,3,4\}$ or $\{5,6,7\}$ are on a 3-line through 12 (otherwise, the other four 3-lines cannot be completed).

Hence it is always possible to apply the following operation: Change one of the two points 1 into 12 on a 4-line. Then change those two points 12 into 1 on the 3-lines which are not connected to a point collinear with 1 on a 4-line. After suitable changes of the 2-lines this yields a $(r,1)$-design with parameters $v=12$, $r=7$, $d_5=12$ (see 2.1.1).

The reverse operation is described in the following lemma.

Lemma 3.1. *All the $(r,1)$-designs with $v=12, r=7, c_5=1, d_5=10, e_5=1$ can be obtained by 'identifying' 2 points which are not connected to a common point on 4-line or a 3-line.*

Remark 3.2. In graph-theoretic language the statement of the lemma means that in the cubic graphs with one-factorization 2 edges are identified which belong to different 1-factors and do not have a common neighbour edge.

To identify points a and b means that both points are called a if they occur on a 4-line and b if they occur on a 3-line or vice versa.

In order to apply this method and to avoid superflous constructions some automorphisms of the $(r,1)$-designs with $v=12$, $r=7$, $d_5=12$ are given. The obtained orbits are considered in the constructions below. For each orbit a representative is used to find all possible pairs of points (edges) which can be identified.

— no. **1**: $\alpha=(1)(2,4)(3,5)(6)(7)(8,9)(10,11)(12)$, $\beta=(1)(2,3)(4,5)(6,7)(8,10)(9,11)(12)$, $\gamma=(1,12)(2,8)(3,10)(4,9)(5,11)(6)(7)$. 2 pairs: 1–8 and 2–10.

— no. **2**: $\alpha=(1)(2,4)(3,5)(6)(7)(8,9)(10,11)(12)$, $\beta=(1,12)(2,11)(3,9)(4,10)(5,8)(6)(7)$. Again 2 pairs: 1–10 and 2–9.

— no. **3**: $\alpha=(1,2,4)(3,6,5)(7,11,9),(8,12,10)$, $\beta=(1)(2,4)(3,5)(6)(7,10)(8,9)(11,12)$. Pair 1–11.

— no. **4**: $\alpha=(1,8)(2,9)(3)(4,11)(5)(7)(10)$, $\beta=(1,6)(2)(3,7)(4)(5,10)(8,12)(9)(11)$, $\gamma=(1)(2,5)(3,4)(6,8)(7,11)(9,10)(12)$. Pair 2–11.

— no. **6**: $\alpha=(1,8)(2,11)(3,5)(4,9)(6,12)(7)(10)$, $\beta=(1)(2,4)(3,5)(6)(7,10)(8)(9,11)(12)$. Pair 1–11.

— no. **8**: $\alpha=(1)(2,3)(4,5)(6)(7)(8,9)(10,11)(12)$, $\beta=(1)(2,4)(3,5)(6)(7)(8,11)(9,10)(12)$. Pair 1–8.

In the case of designs no. **5** and no. **7** no automorphisms are given since there is no pair at all the points of which can be identified.

The 8 pairs given above produce 8 designs with the new set of parameters. In order to decide whether these 8 designs are non-isomorphic regard them as graphs with 3 colours. The points are edges and the lines are vertices; those points which occur 4 times on a 3-line are neglected. The 4-lines represent the 3-colours.

Now regard the number of triangles in these graphs. Design no. **2** is the only one with 2 triangles. The designs no. **1** and **4** have 1 triangle each, but the colours in the unique 4-cycles do not agree. In design no. **1** both pairs of 'parallel' edges are coloured equally which is not the case in design no. **4**. Design no. **8** is the only disconnected design.

The remaining designs no. **3**, **5**, **6**, and **7** have to be discussed more carefully since they all do not contain a triangle. Similar arguments as above show that they are nonisomorphic.

Acknowledgement

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